

PROBABILITY, MARKOV CHAINS, QUEUES, AND SIMULATION

The Mathematical Basis of
Performance Modeling

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