

Monique Jeanblanc • Marc Yor •> Marc Chesney

# Mathematical Methods for Financial Markets

4y Springer

# Contents

## Part I Continuous Path Processes

<b>Continuous-Path Random Processes: Mathematical Prerequisites.</b>	3
1.1 Some Definitions . . . . .	3
1.1.1 Measurability . . . . .	3
1.1.2 Monotone Class Theorem . . . . .	4
1.1.3 Probability Measures . . . . .	5
1.1.4 Filtration . . . . .	5
1.1.5 Law of a Random Variable, Expectation . . . . .	6
1.1.6 Independence . . . . .	6
1.1.7 Equivalent Probabilities and Radon-Nikodym Densities . . . . .	7
1.1.8 Construction of Simple Probability Spaces . . . . .	8
1.1.9 Conditional Expectation . . . . .	9
1.1.10 Stochastic Processes . . . . .	10
1.1.11 Convergence . . . . .	12
1.1.12 Laplace Transform . . . . .	13
1.1.13 Gaussian Processes . . . . .	15
1.1.14 Markov Processes . . . . .	15
1.1.15 Uniform Integrability . . . . .	18
1.2 Martingales . . . . .	19
1.2.1 Definition and Main Properties . . . . .	19
1.2.2 Spaces of Martingales . . . . .	21
1.2.3 Stopping Times . . . . .	21
1.2.4 Local Martingales . . . . .	25
1.3 Continuous Semi-martingales . . . . .	27
1.3.1 Brackets of Continuous Local Martingales . . . . .	27
1.3.2 Brackets of Continuous Semi-martingales . . . . .	29
1.4 Brownian Motion . . . . .	30
1.4.1 One-dimensional Brownian Motion . . . . .	30
1.4.2 d-dimensional Brownian Motion . . . . .	34

## Contents

1.4.3	Correlated Brownian Motions . . . . .	34
1.5	Stochastic Calculus. . . . .	35
1.5.1	Stochastic Integration . . . . .	36
1.5.2	Integration by Parts. . . . .	38
1.5.3	Ito's Formula: The Fundamental Formula of Stochastic Calculus. . . . .	38
1.5.4	Stochastic Differential Equations. . . . .	43
1.5.5	Stochastic Differential Equations: The One-dimensional Case. . . . .	47
1.5.6	Partial Differential Equations. . . . .	51
1.5.7	Doleans-Dade Exponential. . . . .	52
1.6	Predictable Representation Property. . . . .	55
1.6.1	Brownian Motion Case. . . . .	55
1.6.2	Towards a General Definition of the Predictable Representation Property. . . . .	57
1.6.3	Dudley's Theorem; . . . . .	60
1.6.4	Backward Stochastic Differential Equations . . . . .	61
1.7	Change of Probability and Girsanov's Theorem. . . . .	66
1.7.1	Change of Probability. . . . .	66
1.7.2	Decomposition of P-Martingales as Q-semi-martingales . . . . .	68
1.7.3	Girsanov's Theorem: The One-dimensional Brownian Motion Case. . . . .	69
1.7.4	Multidimensional Case. . . . .	72
1.7.5	Absolute Continuity. . . . .	73
1.7.6	Condition for Martingale Property of Exponential Local Martingales . . . . .	74
1.7.7	Predictable Representation Property under a Change of Probability. . . . .	77
1.7.8	An Example of InVariance of BM under Change of Measure . . . . .	78
	<b>Basic Concepts and Examples in Finance . . . . .</b>	79
2.1	A Semi-martingale Framework . . . . .	79
2.1.1	The Financial Market . . . . .	80
2.1.2	Arbitrage Opportunities. . . . .	83
2.1.3	Equivalent Martingale Measure . . . . .	85
2.1.4	Admissible Strategies. . . . .	85
2.1.5	Complete Market . . . . .	87
2.2	A Diffusion Model . . . . .	89
2.2.1	Absence of Arbitrage. . . . .	90
2.2.2	Completeness of the Market . . . . .	90
2.2.3	PDE Evaluation of Contingent Claims in a Complete Market . . . . .	92
2.3	The Black and Scholes Model. . . . .	93
2.3.1	The Model . . . . .	94

2.3.2	European Call and Put Options . . . . .	97
2.3.3	The Greeks . . . . .	101
2.3.4	General Case . . . . .	102
2.3.5	Dividend Paying Assets . . . . .	102
2.3.6	Role of Information . . . . .	104
2.4	Change of Numeraire . . . . .	105
2.4.1	Change of Numeraire and Black-Scholes Formula . . . . .	106
2.4.2	Self-financing Strategy and Change of Numeraire . . . . .	107
2.4.3	Change of Numeraire and Change of Probability . . . . .	108
2.4.4	Forward Measure . . . . .	108
2.4.5	Self-financing Strategies: Constrained Strategies . . . . .	109
2.5	Feynman-Kac- . . . . .	112
2.5.1	Feynman-Kac Formula . . . . .	112
2.5.2	Occupation Time for a Brownian Motion . . . . .	113
2.5.3	Occupation Time for a Drifted Brownian Motion . . . . .	114
2.5.4	Cumulative Options. . . . .	116
2.5.5	Quantiles. . . . .	118
2.6	Ornstein-Uhlenbeck Processes and Related Processes . . . . .	119
2.6.1	Definition and Properties . . . . .	119
2.6.2	Zero-coupon Bond . . . . .	123
2.6.3	Absolute Continuity Relationship for Generalized Vasicek Processes . . . . .	124
2.6.4	Square of a Generalized Vasicek Process . . . . .	127
2.6.5	Powers of $S$ -Dimensional Radial OU Processes, Alias CIR Processes . . . . .	128
2.7	Valuation of European Options . . . . .	129
2.7.1	The Garman and Kohlhagen Model for Currency Options . . . . .	129
2.7.2	Evaluation of an Exchange Option . . . . .	130
2.7.3	Quanto Options . . . . .	132
<b>Hitting Times: A Mix of Mathematics and Finance</b>	. . . . .	135
3.1	Hitting Times and the Law of the Maximum for Brownian Motion . . . . .	136
3.1.1	The Law of the Pair of Random Variables $(W_t, M_t)$ . . . . .	136
3.1.2	Hitting Times Process . . . . .	138
3.1.3	Law of the Maximum of a Brownian Motion over $[0, t]$ . . . . .	139
3.1.4	Laws of Hitting Times . . . . .	140
3.1.5	Law of the Infimum . . . . .	142
3.1.6	Laplace Transforms of Hitting Times . . . . .	143
3.2	Hitting Times for a Drifted Brownian Motion . . . . .	145
3.2.1	Joint Laws of $(M^x, X)$ and $(m^x, X)$ at Time $t$ . . . . .	145
3.2.2	Laws of Maximum, Minimum, and Hitting Times . . . . .	147
3.2.3	Laplace Transforms . . . . .	148
3.2.4	Computation of $W^M(I_{\{T_i/(x) < t\}} e^{-Ar_*} W)$ . . . . .	149

## Contents

3.2.5	Normal Inverse Gaussian Law . . . . .	150
3.3	Hitting Times for Geometric Brownian Motion . . . . .	151
3.3.1	Laws of the Pairs $(M^?, S_t)$ and $\{mf, S_t\}$ . . . . .	151
3.3.2	Laplace Transforms . . . . .	152
3.3.3	Computation of $E(e^{-AT''}(^5)1_{\{T_0(s)<t\}})$ . . . . !	153
3.4	Hitting Times in Other Cases . . . . .	153
3.4.1	Ornstein-Uhlenbeck Processes . . . . .	153
3.4.2	Deterministic Volatility and Nonconstant Barrier . . . . .	154
3.5	Hitting Time of a Two-sided Barrier for BM and GBM . . . . .	156
3.5.1	Brownian Case . . . . .	156
3.5.2	Drifted Brownian Motion . . . . .	159
3.6	Barrier Options . . . . .	160
3.6.1	Put-Call Symmetry . . . . .	160
3.6.2	Binary Options and A's . . . . .	163
3.6.3	Barrier Options: General Characteristics . . . . .	164
3.6.4	Valuation and Hedging of a Regular Down-and-In Call Option When the Underlying is a Martingale . . . . .	166
3.6.5	Mathematical Results Deduced from the Previous Approach . . . . .	169
3.6.6	Valuation and Hedging of Regular Down-and-In Call Options: The General Case . . . . .	172
3.6.7	Valuation and Hedging of Reverse Barrier Options . . . . .	175
3.6.8	The Emerging Calls Method . . . . .	177
3.6.9	Closed Form Expressions . . . . .	178
3.7	Lookback Options . . . . .	179
3.7.1	Using Binary Options . . . . .	179
3.7.2	Traditional Approach . . . . .	180
3.8	Double-barrier Options . . . . .	182
3.9	Other Options . . . . .	183
3.9.1	Options Involving a Hitting Time . . . . .	183
3.9.2	Boost Options . . . . .	184
3.9.3	Exponential Down Barrier Option . . . . .	186
3.10	A Structural Approach to Default Risk . . . . .	188
3.10.1	Merton's Model . . . . .	188
3.10.2	First Passage Time Models . . . . .	190
3.11	American Options . . . . .	191
3.11.1	American Stock Options . . . . .	192
3.11.2	American Currency Options . . . . .	193
3.11.3	Perpetual American Currency Options . . . . .	195
3.12	Real Options . . . . .	198
3.12.1	Optimal Entry with Stochastic Investment Costs . . . . .	198
3.12.2	Optimal Entry in the Presence of Competition . . . . .	201
3.12.3	Optimal Entry and Optimal Exit . . . . .	204
3.12.4	Optimal Exit and Optimal Entry in the Presence of Competition . . . . .	205

3.12.5 Optimal Entry and Exit Decisions . . . . .	206
<b>Complements on Brownian Motion . . . . .</b>	<b>211</b>
4.1 Local Time . . . . .	211
4.1.1 A Stochastic Fubim Theorem . . . . .	211
4.1.2 Occupation Time Formula . . . . .	211
4.1.3 An Approximation of Local Time . . . . .	213
4.1.4 Local Times for Semi-martingales . . . . .	214
4.1.5 Tanaka's Formula . . . . .	214
4.1.6 The Balayage Formula . . . . .	216
4.1.7 Skorokhod's Reflection Lemma . . . . .	217
4.1.8 Local Time of a Semi-martingale . . . . .	222
4.1.9 Generalized Ito-Tanaka Formula . . . . .	226
4.2 Applications . . . . .	227
4.2.1 Dupire's Formula . . . . .	227
4.2.2 Stop-Loss Strategy . . . . .	229
4.2.3 Knock-out BOOST . . . . .	230
4.2.4 Passport Options . . . . .	232
4.3 Bridges, Excursions, and Meanders . . . . .	232
4.3.1 Brownian Motion Zeros . . . . .	232
4.3.2 Excursions . . . . .	232
4.3.3 Laws of $T_x$ , $d_t$ and $g_t$ . . . . .	233
4.3.4 Laws of $(B_b, g_b, d_t)$ . . . . .	236
4.3.5 Brownian Bridge . . . . .	237
4.3.6 Slow Brownian Filtrations . . . . .	241
4.3.7 Meanders . . . . .	242
4.3.8 The Azema Martingale . . . . .	243
4.3.9 Drifted Brownian Motion . . . . .	244
4.4 Parisian Options . . . . .	246
4.4.1 The Law of $(G_{\sim D} \setminus W), W_{G^-}, i$ . . . . .	249
4.4.2 Valuation of a Down-and-In Parisian Option . . . . .	252
4.4.3 PDE Approach . . . . .	256
4.4.4 American Parisian Options . . . . .	257
<b>Complements on Continuous Path Processes . . . . .</b>	<b>259</b>
5.1 Time Changes . . . . .	259
5.1.1 Inverse of an Increasing Process . . . . .	259
5.1.2 Time Changes and Stopping Times . . . . .	260
5.1.3 Brownian Motion and Time Changes . . . . .	261
5.2 Dual Predictable Projections . . . . .	264
5.2.1 Definitions . . . . .	264
5.2.2 Examples . . . . .	266
0.3 Diffusions . . . . .	269
5.3.1 (Time-homogeneous) Diffusions . . . . .	270
5.3.2 Scale Function and Speed Measure . . . . .	270

## Contents

5.3.3	Boundary Points . . . . .	273
5.3.4	Change of Time or Change of Space Variable . . . . .	275
5.3.5	Recurrence . . . . .	277
5.3.6	Resolvent Kernel and Green Function . . . . .	277
5.3.7	Examples . . . . .	279
5.4	Non-homogeneous Diffusions . . . . .	281
5.4.1	Kolmogorov's Equations . . . . .	281
5.4.2	Application: Dupire's Formula . . . . .	284
5.4.3	Fokker-Planck Equation . . . . .	286
5.4.4	Valuation of Contingent Claims . . . . .	289
5.5	Local Times for a Diffusion . . . . .	290
5.5.1	Various Definitions of Local Times . . . . .	290
5.5.2	Some Diffusions Involving Local Time . . . . .	291
5.6	Last Passage Times . . . . .	294
5.6.1	Notation and Basic Results . . . . .	294
5.6.2	Last Passage Time; of a Transient Diffusion . . . . .	294
5.6.3	Last Passage Time Before Hitting a Level . . . . .	297
5.6.4	Last Passage Time Before Maturity . . . . .	298
5.6.5	Absolutely Continuous Compensator . . . . .	301
5.6.6	Time When the Supremum is Reached . . . . .	302
5.6.7	Last Passage Times for Particular Martingales . . . . .	303
5.7	Pitman's Theorem about $(2M_t - W_t)$ . . . . .	306
5.7.1	Time Reversal of Brownian Motion . . . . .	306
5.7.2	Pitman's Theorem . . . . .	307
5.8	Filtrations . . . . .	309
5.8.1	Strong and Weak Brownian Filtrations . . . . .	310
5.8.2	Some Examples . . . . .	312
5.9	Enlargements of Filtrations . . . . .	315
5.9.1	Immersion of Filtrations . . . . .	315
5.9.2	The Brownian Bridge as an Example of Initial Enlargement . . . . .	318
5.9.3	Initial Enlargement: General Results . . . . .	319
5.9.4	Progressive Enlargement . . . . .	323
5.10	Filtering the Information . . . . .	329
5.10.1	Independent Drift . . . . .	329
5.10.2	Other Examples of Canonical Decomposition . . . . .	330
5.10.3	Innovation Process . . . . .	331
<b>A Special Family of Diffusions: Bessel Processes</b> . . . . .		333
6.1	Definitions and First Properties . . . . .	333
6.1.1	The Euclidean Norm of the n-Dimensional Brownian Motion . . . . .	333
6.1.2	General Definitions . . . . .	334
6.1.3	Path Properties . . . . .	337
6.1.4	Infinitesimal Generator . . . . .	337

6.1.5	Absolute Continuity . . . . .	339
6.2	Properties . . . . .	342
6.2.1	Additivity of BESQ's . . . . .	342
6.2.2	Transition Densities . . . . .	343
6.2.3	Hitting Times for Bessel Processes . . . . .	345
6.2.4	Lamperti's Theorem . . . . .	347
6.2.5	Laplace Transforms . . . . .	349
6.2.6	BESQ Processes with Negative Dimensions . . . . .	353
6.2.7	Squared Radial Ornstein-Uhlenbeck . . . . .	356
6.3	Cox-Ingersoll-Ross Processes . . . . .	356
6.3.1	CIR Processes and BESQ . . . . .	357
6.3.2	Transition Probabilities for a CIR Process . . . . .	358
6.3.3	CIR Processes as Spot Rate Models . . . . .	359
6.3.4	Zero-coupon Bond . . . . .	361
6.3.5	Inhomogeneous CIR Process . . . . .	364
6.4	Constant Elasticity of Variance Process . . . . .	365
6.4.1	Particular Case $\mu = 0$ . . . . .	366
6.4.2	CEV Processes and CIR Processes . . . . .	368
6.4.3	CEV Processes and BESQ Processes . . . . .	368
6.4.4	Properties . . . . .	370
6.4.5	Scale Functions for CEV Processes . . . . .	371
6.4.6	Option Pricing in a CEV Model . . . . .	372
6.5	Some Computations on Bessel Bridges . . . . .	373
6.5.1	Bessel Bridges . . . . .	373
6.5.2	Bessel Bridges and Ornstein-Uhlenbeck Processes . . . . .	374
6.5.3	European Bond Option . . . . .	376
6.5.4	American Bond Options and the CIR Model . . . . .	378
6.6	Asian Options . . . . .	381
6.6.1	Parity and Symmetry Formulae . . . . .	382
6.6.2	Laws of $A^{\rho_1}$ and $A^{\lambda}$ . . . . .	383
6.6.3	The Moments of $A_t$ . . . . .	388
6.6.4	Laplace Transform Approach . . . . .	389
6.6.5	PDE Approach . . . . .	391
6.7	Stochastic Volatility . . . . .	392
6.7.1	Black and Scholes Implied Volatility . . . . .	392
6.7.2	A General Stochastic Volatility Model . . . . .	392
6.7.3	Option Pricing in Presence of Non-normality of Returns: The Martingale Approach . . . . .	393
6.7.4	Hull and White Model . . . . .	396
6.7.5	Closed-form Solutions in Some Correlated Cases . . . . .	398
6.7.6	PDE Approach . . . . .	401
6.7.7	Heston's Model . . . . .	401
6.7.8	Mellin Transform . . . . .	403

**Part II Jump Processes**

<b>Default Risk: An Enlargement of Filtration Approach</b>	407
7.1 A Toy Model . . . . .	407
7.1.1 Defaultable Zero-coupon with Payment at Maturity . . . . .	408
7.1.2 Defaultable Zero-coupon with Payment at Hit . . . . .	410
7.2 Toy Model and Martingales . . . . .	412
7.2.1 Key Lemma . . . . .	412
7.2.2 The Fundamental Martingale . . . . .	412
7.2.3 Hazard Function . . . . .	413
7.2.4 Incompleteness of the Toy Model, non Arbitrage Prices . . . . .	415
7.2.5 Predictable Representation Theorem . . . . .	415
7.2.6 Risk-neutral Probability Measures . . . . .	416
7.2.7 Partial Information: Duffie and Lando's Model . . . . .	418
7.3 Default Times with a Given Stochastic Intensity . . . . .	418
7.3.1 Construction of Default Time with a Given Stochastic Intensity . . . . .	418
7.3.2 Conditional Expectation with Respect to $jF_t$ . . . . .	419
7.3.3 Enlargements of Filtrations . . . . .	420
7.3.4 Conditional Expectations with Respect to $Qt$ . . . . .	420
7.3.5 Conditional Expectations of Foo-Measurable Random Variables . . . . .	422
7.3.6 Correlated Defaults: Copula Approach . . . . .	423
7.3.7 Correlated Defaults: Jarrow and Yu's Model . . . . .	425
7.4 Conditional Survival Probability Approach . . . . .	426
7.4.1 Conditional Expectations . . . . .	427
7.5 Conditional Survival Probability Approach and Immersion . . . . .	428
7.5.1 (Ti)-Hypothesis and Arbitrages . . . . .	429
7.5.2 Pricing Contingent Claims . . . . .	430
7.5.3 Correlated Defaults: Kusuoka's Example . . . . .	431
7.5.4 Stochastic Barrier . . . . .	432
7.5.5 Predictable Representation Theorems . . . . .	432
7.5.6 Hedging Contingent Claims with DZC . . . . .	434
7.6 General Case: Without the (W)-Hypothesis . . . . .	437
7.6.1 An Example of Partial Observation . . . . .	437
7.6.2 Two Defaults, Trivial Reference Filtration . . . . .	440
7.6.3 Initial Times . . . . .	442
7.6.4 Explosive Defaults . . . . .	444
7.7 Intensity Approach . . . . .	445
7.7.1 Definition- . . . . .	445
7.7.2 Valuation Formula . . . . .	446
7.8 Credit Default Swaps . . . . .	446
7.8.1 Dynamics of the CDS's Price in a single name setting . . . . .	447
7.8.2 Dynamics of the CDS's Price in a multi-name setting . . . . .	448

7.9	PDE Approach for Hedging Defaultable Claims . . . . .	449
7.9.1	Defaultable Asset with Total Default . . . . .	449
7.9.2	PDE for Valuation . . . . .	450
7.9.3	General Case . . . . .	454
<b>8</b>	<b>Poisson Processes and Ruin Theory . . . . .</b>	<b>457</b>
8.1	Counting Processes and Stochastic Integrals . . . . .	457
8.2	Standard Poisson Process . . . . .	459
8.2.1	Definition and First Properties . . . . .	459
8.2.2	Martingale Properties . . . . .	461
8.2.3	Infinitesimal Generator . . . . .	464
8.2.4	Change of Probability Measure: An Example . . . . .	465
8.2.5	Hitting Times . . . . .	466
8.3	Inhomogeneous Poisson Processes . . . . .	467
8.3.1	Definition . . . . .	467
8.3.2	Martingale Properties . . . . .	467
8.3.3	Watanabe's Characterization of Inhomogeneous Poisson Processes . . . . .	468
8.3.4	Stochastic Calculus . . . . .	469
8.3.5	Predictable Representation Property . . . . .	473
8.3.6	Multidimensional Poisson Processes . . . . .	474
8.4	Stochastic Intensity Processes . . . . .	475
8.4.1	Doubly Stochastic Poisson Processes . . . . .	475
8.4.2	Inhomogeneous Poisson Processes with Stochastic Intensity . . . . .	476
8.4.3	Ito's Formula . . . . .	476
8.4.4	Exponential Martingales . . . . .	477
8.4.5	Change of Probability Measure . . . . .	478
8.4.6	An Elementary Model of Prices Involving Jumps . . . . .	479
8.5	Poisson Bridges . . . . .	480
8.5.1	Definition of the Poisson Bridge . . . . .	480
8.5.2	Harness Property . . . . .	481
8.6	Compound Poisson Processes . . . . .	483
8.6.1	Definition and Properties . . . . .	483
8.6.2	Integration Formula . . . . .	484
8.6.3	Martingales . . . . .	485
8.6.4	Ito's Formula . . . . .	492
8.6.5	Hitting Times . . . . .	492
8.6.6	Change of Probability Measure . . . . .	494
8.6.7	Price Process . . . . .	495
8.6.8	Martingale Representation Theorem . . . . .	496
8.6.9	Option Pricing . . . . .	497
8.7	Ruin Process . . . . .	497
8.7.1	Ruin Probability . . . . .	497
8.7.2	Integral Equation . . . . .	498

8.7.3	An Example . . . . .	498
8.8	Marked Point Processes . . . . .	501
8.8.1	Random Measure . . . . .	501
8.8.2	Definition . . . . .	501
8.8.3	An Integration Formula . . . . .	503
8.8.4	Marked Point Processes with Intensity and Associated Martingales . . . . .	503
8.8.5	Girsanov's Theorem . . . . .	504
8.8.6	Predictable Representation Theorem . . . . .	504
8.9	Poisson Point Processes . . . . .	505
8.9.1	Poisson Measures . . . . .	505
8.9.2	Point Processes . . . . .	506
8.9.3	Poisson Point Processes . . . . .	506
8.9.4	The Ito Measure of Brownian Excursions . . . . .	507
<b>9</b>	<b>General Processes: Mathematical Facts . . . . .</b>	<b>509</b>
9.1	Some Basic Facts about cadlag Processes . . . . .	509
9.1.1	An Illustrative Lemma . . . . .	509
9.1.2	Finite Variation Processes, Pure Jump Processes . . . . .	510
9.1.3	Some er-algebras . . . . .	512
9.2	Stochastic Integration for Square Integrable Martingales . . . . .	513
9.2.1	Square Integrable Martingales . . . . .	513
9.2.2	Stochastic Integral . . . . .	516
9.3	Stochastic Integration for Semi-martingales . . . . .	517
9.3.1	Local Martingales . . . . .	517
9.3.2	Quadratic Covariation and Predictable Bracket of Two Local Martingales . . . . .	519
9.3.3	Orthogonality . . . . .	521
9.3.4	Semi-martingales . . . . .	522
9.3.5	Stochastic Integration for Semi-martingales . . . . .	524
9.3.6	Quadratic Covariation of Two Semi-martingales . . . . .	525
9.3.7	Particular Cases . . . . .	525
9.3.8	Predictable Bracket of Two Semi-martingales . . . . .	527
9.4	Ito's Formula and Girsanov's Theorem . . . . .	528
9.4.1	Ito's Formula: Optional and Predictable Forms . . . . .	528
9.4.2	Semi-martingale Local Times . . . . .	531
9.4.3	Exponential Semi-martingales . . . . .	532
9.4.4	Change of Probability, Girsanov's Theorem . . . . .	534
9.5	Existence and Uniqueness of the e.m.m . . . . .	537
9.5.1	Predictable Representation Property . . . . .	537
9.5.2	Necessary Conditions for Existence . . . . .	538
9.5.3	Uniqueness Property . . . . .	542
9.5.4	Examples . . . . .	543
9.6	Self-financing Strategies and Integration by Parts . . . . .	544

9.6.1	The Model . . . . .	545
9.6.2	Self-financing Strategies and Change of Numeraire . . . . .	545
9.7	Valuation in an Incomplete Market . . . . .	547
9.7.1	Replication Criteria . . . . .	548
9.7.2	Choice of an Equivalent Martingale Measure . . . . .	549
9.7.3	Indifference Prices . . . . .	550
10	Mixed Processes . . . . .	551
10.1	Definition . . . . .	551
10.2	Ito's Formula . . . . .	552
10.2.1	Integration by Parts . . . . .	552
10.2.2	Ito's Formula: One-dimensional Case . . . . .	553
10.2.3	Multidimensional Case . . . . .	555
10.2.4	Stochastic Differential Equations . . . . .	556
10.2.5	Feynman-Kac Formula . . . . .	557
10.2.6	Predictable Representation Theorem . . . . .	558
10.3	Change of Probability !! . . . . .	559
10.3.1	Exponential Local Martingales . . . . .	559
10.3.2	Girsanov's Theorem . . . . .	560
10.4	Mixed Processes in Finance . . . . .	561
10.4.1	Computation of the Moments . . . . .	561
10.4.2	Symmetry . . . . .	562
10.4.3	Hitting Times . . . . .	563
10.4.4	Affine Jump-Diffusion Model . . . . .	565
10.4.5	General Jump-Diffusion Processes . . . . .	569
10.5	Incompleteness . . . . .	569
10.5.1	The Set of Risk-neutral Probability Measures . . . . .	570
10.5.2	The Range of Prices for European Call Options . . . . .	572
10.5.3	General Contingent Claims . . . . .	575
10.6	Complete Markets with Jumps . . . . .	578
10.6.1	A Three Assets Model . . . . .	578
10.6.2	Structure Equations . . . . .	579
10.7	Valuation of Options . . . . .	582
10.7.1	The Valuation of European Options . . . . .	584
10.7.2	American Option . . . . .	586
11	Levy Processes . . . . .	591
11.1	Infinitely Divisible Random Variables . . . . .	592
11.1.1	Definition . . . . .	592
11.1.2	Self-decomposable Random Variables . . . . .	596
11.1.3	Stable Random Variables . . . . .	598
11.2	Levy Processes . . . . .	599
11.2.1	Definition and Main Properties . . . . .	599
11.2.2	Poisson Point Processes, Levy Measures . . . . .	601
11.2.3	Levy-Khintchine Formula for a Levy Process . . . . .	606

## Contents

11.2.4 Ito's Formulae for a One-dimensional Levy Process . . . . .	612
11.2.5 Ito's Formula for Levy-Ito Processes . . . . .	613
11.2.6 Martingales . . . . .	615
11.2.7 Harness Property . . . . .	620
11.2.8 Representation Theorem of Martingales in a Levy Setting . . . . .	621
11.3 Absolutely Continuous Changes of Measures . . . . .	623
11.3.1 Esscher Transform . . . . .	623
11.3.2 Preserving the Levy Property with Absolute Continuity	625
11.3.3 General Case . . . . .	627
11.4 Fluctuation Theory . . . . .	628
11.4.1 Maximum and Minimum . . . . .	628
11.4.2 Pecherskii-Rogozin Identity . . . . .	631
11.5 Spectrally Negative Levy Processes . . . . .	632
11.5.1 Two-sided Exit Times . . . . .	632
11.5.2 Laplace Exponent of the Ladder Process . . . . .	633
11.5.3 D. Kendall's Identity . . . . .	633
11.6 Subordinators . . . . .	634
11.6.1 Definition and Examples . . . . .	634
11.6.2 Levy Characteristics of a Subordinated Process . . . . .	636
11.7 Exponential Levy Processes as Stock Price Processes . . . . .	636
11.7.1 Option Pricing with Esscher Transform . . . . .	636
11.7.2 A Differential Equation for Option Pricing . . . . .	637
11.7.3 Put-call Symmetry . . . . .	638
11.7.4 Arbitrage and Completeness . . . . .	639
11.8 Variance-Gamma Model . . . . .	639
11.9 Valuation of Contingent Claims . . . . .	641
11.9.1 Perpetual American Options . . . . .	641
<b>List of Special Features, Probability Laws, and Functions . . . . .</b>	<b>647</b>
A.I Main Formulae . . . . .	647
A.1.1 Absolute Continuity Relationships . . . . .	647
A.I.2 Bessel Processes . . . . .	648
A.1.3 Brownian Motion . . . . .	649
A.1.4 Diffusions . . . . .	650
A.1.5 Finance . . . . .	650
A.1.6 Girsanov's Theorem . . . . .	651
A.I.7 Hitting Times . . . . .	651
A.I.8 Ito's Formulae . . . . .	651
A.1.9 Levy Processes . . . . .	653
A.1.10 Semi-martingales . . . . .	654
A.2 Processes . . . . .	655
A.3 Some Main Models . . . . .	655
A.4 Some Important Probability Distributions . . . . .	656
A.4.1 Laws with Density . . . . .	656

A.4.2	Some Algebraic Properties for Special r.v.'s . . . . .	656
A.4.3	Poisson Law . . . . .	657
A.4.4	Gamma, and Inverse Gaussian Law . . . . .	658
A.4.5	Generalized Inverse Gaussian and Normal Inverse Gaussian . . . . .	659
A.4.6	Variance Gamma VG(cr, u, 9) . . . . .	661
A.4.7	Tempered Stable TS(F $\pm$ , C $\pm$ , M $\pm$ ) . . . . .	661
A.5	Special Functions . . . . .	662
A.5.1	Gamma and Beta Functions . . . . .	662
A.5.2	Bessel Functions . . . . .	662
A.5.3	Hermite Functions . . . . .	663
A.5.4	Parabolic Cylinder Functions . . . . .	663
A.5.5	Airy Function . . . . .	663
A.5.6	Rummer Functions . . . . .	664
A.5.7	Whittaker Functions . . . . .	664
A.5.8	Some Laplace Transforms . . . . .	665
	<b>References . . . . .</b>	667
<b>B</b>	<b>Some Papers and Books on Specific Subjects . . . . .</b>	709
B.I	Theory of Continuous Processes . . . . .	709
B.I.1	Books . . . . .	709
B.I.2	Stochastic Differential Equations . . . . .	709
B.I.3	Backward SDE . . . . .	709
B.I.4	Martingale Representation Theorems . . . . .	710
B.I.5	Enlargement of Filtrations . . . . .	710
B.I.6	Exponential Functionals . . . . .	710
B.I.7	Uniform Integrability of Martingales . . . . .	710
B.2	Particular Processes . . . . .	710
B.2.1	Ornstein-Uhlenbeck Processes . . . . .	710
B.2.2	CIR Processes . . . . .	710
B.2.3	CEV Processes . . . . .	711
B.2.4	Bessel Processes . . . . .	711
B.3	Processes with Discontinuous Paths . . . . .	711
B.3.1	Some Books . . . . .	711
B.3.2	Survey Papers . . . . .	711
B.4	Hitting Times . . . . .	711
B.5	Levy Processes . . . . .	712
B.5.1	Books . . . . .	712
B.5.2	Some Papers . . . . .	712
B.6	Some Books on Finance . . . . .	712
B.6.1	Discrete Time . . . . .	712
B.6.2	Continuous Time . . . . .	712
B.6.3	Collective Books . . . . .	712
B.6.4	History . . . . .	713

B.7 Arbitrage . . . . .	713
B.8 Exotic Options . . . . .	713
B.8.1 Books . . . . .	713
B.8.2 Articles . . . . .	713
<b>Index of Authors . . . . .</b>	<b>715</b>
<b>Index of Symbols . . . . .</b>	<b>723</b>
<b>Subject Index . . . . .</b>	<b>725</b>