

# Exercises in Probability

A Guided Tour from Measure Theory to Random Processes,  
via Conditioning

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